

\mathcal{C}^2 Formulation of Euler Fluid

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Abstract

The Hamiltonian formalism for the continuous media is constructed using the representation of Euler variables in $\mathcal{C}^2 \times \infty$ phase space.

1 Introduction

In the theory of continuous media, such as fluid, gas or plasma there exist two kinds of description usually refereed as Lagrange and Euler. The first uses the trajectories of the particles which constitute the media, while in Euler description the role of dynamical variables is played by the velocity $\vec{v}(x, t)$ and density $\rho(x, t)$. As it was discussed in [4] the Lagrange approach which uses the coordinates of the particles which constitute the media, is very convenient to introduce the interaction between the particles and for the construction of Hamiltonian formalism, which looks like usual field theory canonical formalism and the only problem is to construct the x -dependent canonical variables. This problem was solved in [4] and the outcome of the approach suggested there was a mechanical system where evolution is described by canonical Hamiltonian equations in the $2d \times \infty$ -dimensional phase space (for the d -dimensional fluid). Euler variables could be expressed via the canonical variables and algebra of its Poisson brackets has the following form for any dimension of space:

$$\begin{aligned} \{v_j(x), v_k(y)\} &= -\frac{1}{m\rho(x)} (\nabla_j v_k(x) - \nabla_k v_j(x)) \delta(\vec{x} - \vec{y}) \\ \{v_j(x), \rho(y)\} &= \frac{1}{m} \nabla_j \delta(\vec{x} - \vec{y}) \\ \{\rho(x), \rho(y)\} &= 0, \end{aligned} \tag{1}$$

The parameter m here is the mass of the particles which constitute the fluid. Usually it does not appear in Euler description but here it was inherited from

Lagrange description, we put it equal to unity. The relation of Lagrange and Euler description in fluid dynamics was discussed in many text books and articles (see [9], there could be found numerous references to the earlier investigations, [7],[2],[8][4],[1],[3]) and we will not discuss it here. The typical Hamiltonian of ideal fluid could be expressed in terms of Euler variables and it has the following form:

$$H = \int d^3x [\frac{1}{2}m\rho(x)\vec{v}(x)^2 + V(\rho(x))], \quad (2)$$

where the function $V(\rho(x_i))$ describes the "potential" energy of the fluid and could be chosen phenomenologically [6]:

$$V(\rho(x)) = \frac{\kappa}{2\rho_0}(\delta\rho(x))^2 + \lambda(\nabla\rho(x))^2 + \dots, \quad (3)$$

where $\delta\rho(x)$ is the deviation of the density from its homogeneous distribution ρ_{as} at infinity:

$$\delta\rho(x) = \rho(x) - \rho_{as}. \quad (4)$$

The first term in (3) is responsible for the sound wave in the fluid (κ is the velocity of sound), the second term in (3) describes the dispersion of the sound waves. The equations of motion for the variables $\vec{v}(x, t), \rho(x, t)$ could be written in a canonical form:

$$\begin{aligned} \dot{\vec{v}}(x, t) &= \{H, \vec{v}(x, t)\}, \\ \dot{\rho}(x, t) &= \{H, \rho(x, t)\}. \end{aligned} \quad (5)$$

The Poisson brackets to used in (5) are given by (1). The situation we have described looks quite standard for a mechanical system, but the point is that the variables $\vec{v}(x, t), \rho(x, t)$ do not belong to the phase space because the Poisson brackets (1) are degenerate and we can not consider it as usual coordinates of the phase space. This kind of Poisson algebra could be treated by Kirillov-Konstant approach [10]. The center of the algebra (1) depends of the dimension of configuration space. For example, in 2-dimensional case the center is formed by:

$$I_n = \int d^2x \rho(x)^{1-n} (\partial_1 v_2(x) - \partial_2 v_1(x))^n, \quad (6)$$

see [2], [7] for discussion. The 3-dimensional case we shall consider in details below. Here one of the Casimirs is the "helicity" functional

$$Q = \int d^3x \epsilon_{jkl} v_j(x) \nabla_k v_l(x). \quad (7)$$

The other Casimir is the total number of particles N (valid for any dimension, for $d = 2$, $N = I_0$):

$$N = \int d^3x \rho(x) \quad (8)$$

2 C^2 Hydrodynamics

The 3-dimensional case, which is very important for applications was considered by many authors starting from XIX century. It is hardly possible to give an exhaustive list of references. Recently it was discussed in [7] (see also [9] for earlier references) where it was suggested to build Hamiltonian formalism for 3-dimensional Euler fluid using Clebsh parameterization [5] for the velocity :

$$\vec{v}(x, t) = \vec{\partial}\alpha(x, t) + \beta\vec{\partial}\gamma(x, t) \quad (9)$$

where the new functions $\alpha(x, t), \beta(x, t), \gamma(x, t)$ together with density $\rho(x, t)$ are used for the construction of the coordinates of the phase space. What we are going to suggest here is an alternative approach, which has certain advantages.

Let us consider a mechanical system which is described by a pair of complex coordinates which belong to $C^2 \times \infty$: $u_\alpha(x, t)$, $\bar{u}_\alpha(x, t)$, where $\alpha = 1, 2$. The Lagrangian for this system we shall take in the following form:

$$\begin{aligned} L = & \int d^3x \left\{ \frac{im}{2} (\bar{u}(x, t) \dot{u}(x, t) - \dot{\bar{u}}(x, t) u(x, t)) \right. \\ & \left. + m \frac{(\bar{u}(x, t) \vec{\partial} u(x, t) - \vec{\partial} \bar{u}(x, t) u(x, t))^2}{8\bar{u}(x, t) u(x, t)} - V(\bar{u}(x, t) u(x, t)) \right\}, \quad (10) \end{aligned}$$

where we assume the summation over indexes. The canonical momenta, corresponding to the variables $u_\alpha(x)$, $\bar{u}_\alpha(x)$ are given by equations

$$\begin{aligned} p_\alpha^u(x) &= \frac{im}{2} \bar{u}_\alpha(x), \\ p_\alpha^{\bar{u}}(x) &= -\frac{im}{2} u_\alpha(x). \end{aligned} \quad (11)$$

As it expected for the Lagrangian which is a linear function of velocities, the equations (11) define the constraints on the canonical variables:

$$\begin{aligned} \lambda_\alpha^1(x) &= p_\alpha^u(x) - \frac{im}{2} \bar{u}_\alpha(x) \sim 0, \\ \lambda_\alpha^2(x) &= p_\alpha^{\bar{u}}(x) + \frac{im}{2} u_\alpha(x) \sim 0. \end{aligned} \quad (12)$$

The Poisson brackets of the constraints are non-degenerate

$$\{\lambda_\alpha^1(x), \lambda_\beta^2(y_i)\} = im\delta_{\alpha\beta}\delta(\vec{x} - \vec{y}) \quad (13)$$

and we can use these constraints to eliminate canonical momenta $p_\alpha^u(x), p_\alpha^{\bar{u}}(x)$ using Dirac procedure [13]. The resulting Poisson (Dirac) brackets for the rest of coordinates of the phase space $\tilde{\Gamma}$ are:

$$\begin{aligned} \{u_\alpha(x), \bar{u}_\beta(y_i)\} &= \frac{i}{m}\delta_{\alpha\beta}\delta(\vec{x} - \vec{y}), \\ \{u_\alpha(x), u_\beta(y_i)\} &= 0, \\ \{\bar{u}_\alpha(x), \bar{u}_\beta(y_i)\} &= 0. \end{aligned} \quad (14)$$

The Hamiltonian, corresponding to the Lagrangian (10) has the following form

$$H = \int d^3x \left[-m \frac{(\bar{u}(x)\vec{\partial}u(x) - \vec{\partial}\bar{u}(x)u(x))^2}{8\bar{u}(x)u(x)} + V(\bar{u}(x)u(x)) \right] \quad (15)$$

Now we shall explain why we consider this system. Let us form the following objects:

$$\begin{aligned} \vec{v}(x) &= \frac{1}{2i}(\bar{u}(x)\vec{\partial}u(x) - \vec{\partial}\bar{u}(x)u(x)), \\ \rho(x) &= \bar{u}(x)u(x). \end{aligned} \quad (16)$$

The notations we have used here are not accidental. The point is that if we shall calculate the Poisson brackets for (16), using (14) the result will exactly coincide with (1). The Hamiltonian H , given by (15), being expressed via $\vec{v}(x)$ and $\rho(x)$ will take the following form:

$$H = \int d^3x \left[\frac{1}{2}m\rho(x)\vec{v}(x)^2 + V(\rho(x)) \right], \quad (17)$$

which also coincides with Hamiltonian given by (2).

The equations of motion for the variable $u_\alpha(x), \bar{u}_\alpha(x)$ have the usual form:

$$\begin{aligned} \dot{u}_\alpha(x) &= \{H, u_\alpha(x)\} \\ \dot{\bar{u}}_\alpha(x) &= \{H, \bar{u}_\alpha(x)\} \end{aligned} \quad (18)$$

Apparently, the correct equations of motion, including the continuity equation for variables $\vec{v}(x)$ and $\rho(x)$ follow from (18).

As was mentioned above, the description of the fluid in terms of $u_\alpha(x)$, $\bar{u}_\alpha(x)$ is rather similar to the description which uses Clebsh parameterization. Indeed, these variables could be presented in the following form:

$$u_\alpha(x) = \sqrt{\rho(x)} e^{i\phi(x)/2} \begin{pmatrix} e^{-i\psi(x)/2} \cos \frac{\alpha(x)}{2} \\ e^{i\psi(x)/2} \sin \frac{\alpha(x)}{2} \end{pmatrix} \quad (19)$$

from where we obtain the representation for the velocity through angles $\phi(x)$, $\psi(x)$ and $\alpha(x)$

$$\vec{v}(x) = \frac{1}{2}(\vec{\partial}\phi(x) - \vec{\partial}\psi(x)\cos\alpha(x)) \quad (20)$$

These equation defines the velocity, if Clebsh parameters are known. Also, as is well-known (see e.g.[1],[7]) any differentiable vector field $\vec{v}(x)$ has the local representation (20). In other words, knowing $\vec{v}(x)$, we can construct Clebsh parameters $\alpha(x)$, $\phi(x)$, $\psi(x)$ with some ambiguity. This ambiguity arises as a set of integration constants. In our construction this ambiguity could be understood as follows. The Lagrangian function (10) we consider is invariant with respect to the symmetry group $U(2)$ which acts as follows:

$$u_\alpha(x) \rightarrow \tilde{u}_\alpha(x) = T_{\alpha\beta} u_\beta(x), \quad T^+ T = 1 \quad (21)$$

and according to Noether's theorem the integrals of motion, which are the generators of these transformations are :

$$t^0 = \int d^3x \frac{1}{2} \bar{u}(x) u(x), \quad t^a = \int d^3x \bar{u}(x) \frac{\sigma^a}{2} u(x) \quad (22)$$

The transformations (22) change the Clebsh variables, but does not affect the Euler's variables. So, in particular, the constant shift of the angle $\phi(x)$ is generated by ex-Casimir N , which in $\tilde{\Gamma}$ has lost its status, the generator t^3 shifts the angle $\psi(x)$, the other two generators mix the angles $\psi(x)$ and $\alpha(x)$. So the system described by variables $(u_\alpha(x), \bar{u}_\alpha(x))$ is a Hamiltonian system with symmetry and we can reduce its phase space by procedure given by Souriau [11] and Marsden and Weinstein [12]. The reduced phase is the space, where "live" almost all the Euler variables. The latter means that the procedure of reduction implies fixing the integrals of motion, in particular $t^0 = \frac{N}{2}$ does not anymore belongs to the set of variables.

The only problem we have now is the "helicity" functional, which still is the Casimir and the reduction did not remove it. For finite dimensional

systems the existence of Casimir implies the degeneracy of Poisson brackets. It could be easily seen from the following consideration. By definition the Casimir C should have a vanishing brackets with all variable

$$\{p_k, C\} = 0, \quad \{q_k, C\} = 0 \quad (23)$$

where p_k, q_k are all set of coordinates of the phase space. If the Poisson brackets are non-degenerate, the equations (23) mean that C is a constant. The situation for the infinite dimensional system is different because of existence of so called the functionals with zero variation. Consider for example an infinite dimensional system, which is described by the set of canonical variables $p(x), q(x)$ where $x \in R$. The Poisson brackets are non-degenerate:

$$\{p(x), q(y)\} = \delta(x - y). \quad (24)$$

In this case we can easily construct a nontrivial functional, which will have vanishing Poisson brackets with all variables $p(x), q(x)$. It has the following form:

$$C = \int_{-\infty}^{\infty} dx \frac{p'(x)q(x) - p(x)q'(x)}{p^2(x) + q^2(x)} \quad (25)$$

and has a meaning of a winding number for the phase of the complex variable $a(x) = p(x) + iq(x)$, i.e. C is what physicists used to call topological charge. Note that in order for C to be the Casimir it is not necessary to impose the condition on $a(x)|_{x \rightarrow -\infty} = a(x)|_{x \rightarrow +\infty}$ and compactify R . In this case C will take an integer values and indeed will be the winding number.

The "helicity" functional has the same origin as the functional C in this example. In order to see it let us introduce a unit four vector F_k [15]:

$$\frac{1}{\sqrt{\bar{u}(x)u(x)}} \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix} = \begin{pmatrix} F_1(x) + iF_2(x) \\ F_3(x) + iF_4(x) \end{pmatrix} \quad (26)$$

This four vector maps $S^3 \rightarrow S^3$, provided we impose on the variables $\bar{u}_\alpha(x), u_\alpha(x)$ the asymptotic conditions: $u_\alpha(x) \rightarrow u_\alpha^0$, when $|\vec{x}| \rightarrow \infty$ and compactify R^3 . The helicity functional Q given by (7) could be written in the following form:

$$Q = \frac{1}{3} \int d^3x \epsilon_{abcd} \epsilon_{ijk} F_a \partial_i F_b \partial_j F_c \partial_k F_d, \quad (27)$$

which is the standard representation (up to normalization constant) for the winding number of the map $S^3 \rightarrow S^3$, so called Hopf invariant. Here again

we should note that even if we neglect the asymptotic conditions on $u_\alpha(x)$ together with compactification of R^3 , Q still will be invariant with respect to local variations and therefore will have vanishing Poisson brackets with $\bar{u}_\alpha(x), u_\alpha(x)$. So, the conclusion of this arguments is that for infinite dimensional mechanical systems the existence of Casimirs is not necessary implies the degeneracy of Poisson brackets, provided these Casimirs are related to the geometric properties of the phase space. The helicity functional belongs to this class of "friendly" Casimirs.

The description we presented here is very convenient for different kinds of applications and generalizations. First of all it seems rather convenient for investigation of stability problem, because the second variation of Hamiltonian (15) is terms of the variables $\bar{u}_\alpha(x), u_\alpha(x)$ is quite compact and simple. As the generalizations are concerned, one can consider the "relativization" of the Lagrangian (10), supersymmetric extension of the variables $\bar{u}_\alpha(x), u_\alpha(x)$. Increasing the number of components we can consider the media with internal degrees of freedom et cetera. Also this formulation is very convenient for quantization of fluid. We are planning to present some of these subjects in future publications.

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